A function $f : I \to \mathbb{R}$ is **continuous at** c when

$$\lim_{x \to c} f(x) = f(c).$$

If f is continuous at every $c \in I$, we simply say f is **continuous**.

Problem 1
Show that each of the following statements is false using a counterexample.

1. A continuous, surjective function $f : \mathbb{R} \to \mathbb{R}$ is injective.
2. A continuous, injective function $f : \mathbb{R} \to \mathbb{R}$ is surjective.
3. A bijective function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
4. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f \circ g$ is continuous, then f is continuous.
5. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f \circ g$ is continuous, then g is continuous.
6. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f \circ g$ is continuous, then either f or g is continuous.

Problem 2
Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ and $f(a) > 0$, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (a - \delta, a + \delta)$.

Problem 3
Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, and we have for some $a \in \mathbb{R}$, $f(a) > 0$. Furthermore suppose $(fg)(x) = 0$ for all $x \in \mathbb{R}$. Show there exists some $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$, $g(x) = 0$. *Hint: Use the previous problem.*

Problem 4
Show that if $f : \mathbb{R} \to \mathbb{R}$, then f is continuous if and only if for any open interval (a, b), for all $x \in f^{-1}((a, b))$ there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq f^{-1}((a, b))$.

Problem 5
Exhibit a function $f : I \to \mathbb{R}$ which is:

1. Everywhere discontinuous.
2. Continuous only at 0.
3. Continuous only at integers.
4. Continuous only at irrational numbers.